ON THE DUALITY OF OPTIMAL CONTROL AND TRACKING PROBLEMS

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The methods of [1] are used to investigate the duality of a tracking problem with constantly acting disturbances and a control problem with restricted coordinates. Similar matters are discussed for systems with lag. The set of problems in question was first proposed and discussed in N. N. Krasovskii's seminar at Ural University. The elements of the theory of duality [1, 2] of control and tracking problems are thus applied to a specific class of infinite-dimensional problems.

1. Tracking under constantly acting disturbances. We are given the linear n-dimensional system

$$dx / d\tau = Ax + Bf \tag{1.1}$$

with the constant $n \times n$- and $n \times r$-dimensional matrices A and B and with r-dimensional constantly acting disturbances $f(\tau)$ at the input. We assume that the actual function $f(\tau)$ in (1.1) is not known and that we are given only the estimate $f \in F_1$ for all possible f, i. e. that we are given only a description of the set F_1.

We shall disregard more general estimates, limiting our attention to the following classes of functions f.

1) The set F_1 consists of all the r-vector functions of the space $L[t, \vartheta]$ with a norm bounded by unity,

$$\rho_L(f) = \int_t^\vartheta \gamma[f(\xi)]\, d\xi \leqslant 1 \quad (\gamma \text{ is the finite-dimensional norm of the } r\text{-vector } f) \tag{1.2}$$

2) The set F_1 consists of all r-vector functions of the space $L_2[t, \vartheta]$ forming a bounded convex balanced absorbing set [3] in $L_2[t, \vartheta]$.

Under these conditions the set F_1 is closed in $L_2[t, \vartheta]$. Assigning the corresponding Minkowski functional $\rho_{F_1}^*$ to F_1, we obtain [3] a certian norm for the functions f from $L_2[t, \vartheta]$.

The condition $f \in F_1$ can be rewritten as

$$\rho_{F_1}^*(f) \leqslant 1 \tag{1.3}$$

3) the set $F_1 = \{f(\tau) : f(\tau) = dF(\tau) / d\tau\}$; the $dF(\tau) / d\tau$ in this definition are the generalized derivarives [3] of the restricted-variation functions F satisfying the inequality

$$\rho_V(F) = \int_t^\vartheta \gamma[dF] \leqslant 1 \tag{1.4}$$

The signal

$$z(\tau) = Gx(\tau) \quad (G \text{ is a constant } (m \times n)\text{-matrix}) \tag{1.5}$$

is detected in the interval $[t, \vartheta]$.

We are required to find the linear operation (functional) $\varphi[z] = \zeta_1$ which isolates from any possible signal (r-vector function) $z(t)$ the value ζ_1 which differs least from the required value $\zeta = c'x(\vartheta)$ (c is a known vector and the prime denotes transposition). We seek the operation φ in a prescribed class $\varphi \in \Phi$, identifying $\varphi[z]$ with the function $v(t) \in V$, where the class V clearly corresponds to Φ.

Writing out the solution of (1.1) with the boundary condition $x_\vartheta = x(\vartheta)$, we obtain

$$z(\tau) = GX[\tau, \vartheta]\, x_\vartheta - \int_\tau^\vartheta GX[\tau, \xi]\, Bf(\xi)\, d\xi$$

$$dX[t, \vartheta] / dt = AX[t, \vartheta], \qquad X[t, t] = E$$

In accordance with the standard procedure of tracking theory [1, 2], we assume fulfillment of the zero-bias conditions whereby the required operation $\varphi[z]$ must isolate $\zeta = c'x(\vartheta)$ exactly when $f(\tau) \equiv 0$. This implies fulfillment of the equations

$$\varphi[GX[\tau, \vartheta]x_\vartheta] = c'x_\vartheta, \qquad \varphi[GX[\tau, \vartheta]] = c' \tag{1.6}$$

for any x_ϑ.

From now on we assume that

$$\Phi = \bigcup_\nu \Phi_\nu, \qquad \Phi_\nu = \{\varphi : \|\varphi\| \leqslant \nu\}$$

where $\|\varphi\|$ is some norm in the appropriate function space. Let

$$F_\mu = \{f : \mu^{-1}f \in F_1\}$$

i.e. let F_μ represent the set F_1 expanded $\mu > 0$ times, and

$$F_0 = \bigcup_\mu F_\mu$$

Problem 1.1. For a given $\varepsilon \geqslant 0$ we are to find the optimal φ° from among the operations $\varphi \in \Phi$ satisfying (1.6) which ensures fulfillment of the inequality

$$\max_f |\varphi^\circ[z(\vartheta)] - c'x(\vartheta)| \leqslant \varepsilon \qquad (f \in F_1) \tag{1.7}$$

for any possible signal $z(\tau)$ under the condition $\|\varphi^\circ\| = \min = \nu^\circ$.

Condition (1.7) in Problem 1.1 is equivalent to

$$\max_f |\varphi^\circ[z(\vartheta)] - c'x(\vartheta)| \leqslant \varepsilon\rho[f] \qquad (f \in F_0) \tag{1.8}$$

where in accordance with (1), (2), (3) the norm $\rho[f]$ is either $\rho_L[f]$, or $\rho_{F_1}{}^*[f]$, or $\rho_V{}^*[F]$.

Problem 1.2. To find from among $\varphi \in \Phi_\nu$ the optimal operation φ° which ensures that

$$\min_{\varphi \in \Phi_\nu} \max_{f \in F_1} |\varphi[z(t)] - c'x(\vartheta)| = \varepsilon_0 \tag{1.9}$$

Problem 1.2 is meaningful if $\varepsilon_0 > 0$. Condition (1.9) is equivalent to the definition of the smallest number ε which satisfies condition (1.8) for $\varphi \in \Phi_\nu$.

Note 1.1. The fact that, for example, the norm $\|\varphi^\circ\|$ in Problem 1.1 does not necessarily grow to infinity as $\varepsilon \to 0$ is essential to our formulation. In principle, the problem may be solvable even for $\varepsilon = 0$. System (1.1), (1.2), for which we have $\nu^\circ < \infty$ in the solution of Problem 1.1 for $\varepsilon = 0$, can be called c-trackable under constantly acting disturbances. The effective conditions which ensure this property (see also [4]) will not be discussed in the present paper.

2. Solution of the tracking problem. Let the number $\varepsilon \geqslant 0$ be given. We begin by considering the existence of a not necessarily optimal operation $\varphi[z]$ which satisfies Problem 1.1. From (1.5)–(1.7) we obtain

$$\max_{f\in F_1}\int_t^{\vartheta} v(\xi)\left\{\int_{\xi}^{\vartheta} GX[\xi,\tau]Bf(\tau)\,d\tau\right\}d\xi\leqslant\varepsilon$$

from which we infer that

$$\max_f\int_t^{\vartheta} q(\tau)f(\tau)\,d\tau\leqslant\varepsilon\qquad (f\in F_1)\tag{2.1}$$

$$\int_t^{\tau} v(\xi)GX[\xi,\tau]Bd\xi=q(\tau)\tag{2.2}$$

Let the set F_1 be of the form (1) . Then, considering $q(\tau)$ as elements of the conjugate of the space $L[t,\vartheta]$, we find that condition (2.1) is equivalent to the inequality

$$\text{vrai}\ \max_\tau\gamma^*[q(\tau)]\leqslant\varepsilon,\qquad t=\leqslant\tau\leqslant\vartheta$$

which can be replaced by

$$\max_\tau\gamma^*[q(\tau)]\leqslant\varepsilon,\qquad t\leqslant\tau\leqslant\vartheta\tag{2.3}$$

by virtue of the absolute continuity of $q(\tau)$ (see (2.2)).

Here γ^* is the finite-dimensional conjugate of γ. Identifying Φ with the given class of (possibly generalized) functions $V=\{v\}$, we seek $v\in V$ instead of $\varphi\in\Phi$. With these considerations in mind we transform Problem 1.1 into

Problem 2.1.1. To find the optimal v° for which $\|v^\circ\|=\nu^\circ=\min$ from among the m-vector functions $v\in V$ which solve the problem

$$\int_t^{\vartheta} v(\xi)GX[\xi,\vartheta]Bd\xi=c'\tag{2.4}$$

$$\int_t^{\tau} v(\xi)GX[\xi,\tau]Bd\xi=q(\tau)\tag{2.5}$$

$$\gamma^*[q(\tau)]\leqslant\varepsilon,\qquad t\leqslant\tau\leqslant\vartheta\tag{2.6}$$

We note that (2.6) is a restriction imposed on the instantaneous values of the row-vector $q(\tau)$, so that (2.5), (2.6) can be interpreted as an infinite system of linear functional inequalities. Problem 1.2 can be transformed in the same way.

Problem 2.2.1. To find the optimal v° which ensures that

$$\max_\tau\gamma^*[q(\tau)]=\varepsilon^\circ=\min,\qquad t\leqslant\tau\leqslant\vartheta\tag{2.7}$$

from among the m-vector functions $v\in V_\nu=\{v:\|v\|\leqslant\nu\}$.

Let F_1 be a set of the form (2). By virtue of the properties of F_1 we now infer that the condition $f\in F_1$ is equivalent to [1, 3]

$$\int_t^{\vartheta} h(\tau)f(\tau)\,d\tau\leqslant\rho_{F_1}(h)\tag{2.8}$$

whatever the r-vector row-function from $L_2[t,\vartheta]$. Here $\rho_{F_1}(h)$ is the supporting functional of the set F_1,

$$\rho_{F_1}(h)=\max_f\int_t^{\vartheta} h(\xi)f(\xi)\,d\xi\qquad (f\in F_1)$$

Expression (2.8) enables us to rewrite condition (2.1) as

$$\rho_{F_1}(q)\leqslant\varepsilon\tag{2.9}$$

At the same time, concretizing Problems 1.1, 1.2, we obtain

Problem 2.1.2. To find the optimal v° of minimal norm $\| v^\circ \| = \min$ from among the m-vector functions $v(t) \in V$ satisfying (2.4), (2.5), (2.9).

Problem 2.2.2. To find the optimal v° which ensures that

$$\max_\tau \rho_{F_1}(q) = \varepsilon^\circ = \min_v, \qquad t \leqslant \tau \leqslant \vartheta$$

from among the functions $v \in V_\nu$ satisfying (2.4), (2.5).

Turning finally to Case (3), we note that the set F_1 in this case is regularly convex [5] in the space of r-vector functions with restricted variation conjugate to the r-vector space $C[t, \vartheta]$ with the norm

$$\| v \| = \max_\tau \gamma^* [v(\tau)] \qquad (t \leqslant \tau \leqslant \vartheta)$$

The above property of F_1 implies [6] (as in the case considered in [2]) the equivalence of the relation $f \in F_1$ and the inequality

$$\int_t^\vartheta h(\tau) f(\tau)\, d\tau = \int_t^\vartheta h(\tau)\, dF(\tau) \leqslant \rho_V(h)$$

$$\rho_V(h) = \max_F \int_t^\vartheta h(\tau)\, dF(\tau), \qquad \left\{ F(\tau) : \int_t^\vartheta \gamma [dF(\tau)] \leqslant 1 \right\}$$

From this we infer immediately that (2.1) is equivalent to (2.3), so that Problems 1.1, 1.2 are transformable back into Problems 2.1.1, 2.2.2 in Case (3) for the set F_1 Consideration of Case (3) shows that complementing the class F (1) up to the class F (3), i.e. the class of locally integrable functions to the class of first-order generalized functions [3] does not affect the solvability of Problems 1.1, 1.2.

3. Control with restricted coordinates. Let us consider the linear controlled system

$$dy / d\tau = A_1 y + G_1 u, \qquad p(\tau) = B_1 y(\tau), \qquad t \leqslant \tau \leqslant \vartheta \tag{3.1}$$

with restricted coordinates

$$p(\tau) \in P_\varepsilon \tag{3.2}$$

and the boundary conditions $y(t) = 0,\ y(\vartheta) = c$ (the interval $t \leqslant \tau \leqslant \vartheta$ is fixed). We assume that one of the two following conditions holds for P_ε:

$$1) \qquad P_\varepsilon = \{ p(\tau) = \max_\tau \gamma^* [p(\tau)] \leqslant \varepsilon \} \quad (t \leqslant \tau \leqslant \vartheta) \tag{3.3}$$

$$2) \qquad P_\varepsilon = \{ p(\tau) : \rho_{F_1}(p) \leqslant \varepsilon \} \tag{3.4}$$

Here γ^*, ρ_{F_1} are the same as in (2.3), (2.9). From (3.1), (3.2) we obtain the equations

$$\int_t^\vartheta Y[\vartheta, \xi]\, G_1 u(\xi)\, d\xi = c, \quad \int_t^\tau B_1 Y[\tau, \xi]\, G_1 u(\xi)\, d\xi = p(\tau) \tag{3.5}$$

where

$$dY[\tau, \vartheta] / d\tau = AY[\tau, \vartheta], \qquad Y[\tau, \tau] = E \tag{3.6}$$

and either (3.3) or (3.4) applies.

Let u belong to the class $\quad U = \bigcup_\nu U_\nu, \qquad U_\nu = \{ u : \| u \| \leqslant \nu \}$

where $\| u \|$ is some norm in the appropriate function space.

Problem 3.1.1. To find the optimal u° with the minimal norm $\| u^\circ \| = \nu^\circ = \min$

from among the controls $u(\xi)$ which take system (3.1) under restrictions (3.2), (3.3) either from the state $y(t) = 0$ to the state $y(\vartheta) = c$ or from the state $y(t) = y_\alpha$ to the state $y(\vartheta) = y_\beta$, where

$$y_\beta = Y[\vartheta, \ t]y_\alpha = c \tag{3.7}$$

Problem 3.1.2. This problem is similar to Problem 3.1.1 with (3.4) replacing (3.3).

Problems 3.1.1, 3.1.2 are control problems with restricted coordinates [7–9]. Problems of minimization of the restrictions can be formulated in similar fashion.

Problem 3.2.1. To find the optimal u° which ensures that

$$\max_\tau \gamma^* [p(\tau)] = \min = \varepsilon^\circ \qquad (t \leqslant \tau \leqslant \vartheta) \tag{3.8}$$

from among the controls $u \in V_\nu$ which take system (3.1) from $y(t) = 0$ to $y(\vartheta) = c$.

Problem 3.2.2. This is similar to Problem 3.1.1 with the condition

$$\rho_{F_1}(p) = \min_\nu = \varepsilon^\circ \tag{3.9}$$

replacing (3.7).

Problem 3.1.1 is known as the problem of minimizing the maximum deviation of the controlled system from zero [13].

4. The duality of the optimal control and tracking problems. Let systems (1.1), (1.5) and (3.1) satisfy the conditions

$$A = -A_1', \qquad B = B_1', \qquad G = G_1' \tag{4.1}$$

Then, clearly,

$$X[\xi, \tau] = Y'[\tau, \xi] = e^{A(\xi-\tau)}, \qquad q'(\tau) = p(\tau)$$

which means that Eqs. (2.4), (2.5) are simply Eqs. (3.5), (3.6) transposed, with $v(\xi)$ replacing $u(\xi)$.

It is important to note that the restrictions on $p(\tau) = - q'(\tau)$ are the same in both cases. They are defined by inequalities (2.6), (2.9) in the tracking problem and by inequalities (3.2)–(3.4) in the control problem.

Problems 2.1.1 and 3.1.1, 2.1.2 and 3.1.2 can be made to coincide completely simply by taking the same norms for u and v.

Similarly, Problems 2.2.1 and 3.2.1, 2.2.2 and 3.2.2 coincide under the same restrictions $U_\nu = V_\nu$ We shall call the indicated pairs of coincident problems "conjugate" as in [1]. The following statements are valid.

Theorem 4.1. Let conjugate systems (1.1), (1.5), (3.1), (4.1) be given and let $U_\nu = V_\nu$. Let the optimal tracking problem (with constantly acting disturbances) of the vector $z = c'x(\tau)$ be considered in the form 2.1.1 (2.1.2) for system (1.1), (1.5). Problem 2.1.1 (2.1.2) is then equivalent to the conjugate problem of optimal control with restricted coordinates in the form 3.1.1 (3.2.2) for system (3.1), (4.1) under boundary conditions $y(t)$, $y(\theta)$ satisfying (3.7) and under restriction (3.3) ((3.4)).

Theorem 4.2. Let the optimal tracking problem for the minimum tracking error with constantly acting disturbances be considered in the form 2.1.2 (2.2.2) for system (1.1), (1.5). Problem 2.1.2 (2.2.2) is then equivalent to the conjugate Problem 3.1.2 (3.2.2) for the optimal control which minimizes the restrictions on the coordinates for system (3.1), (4.1).

Note 4.1. The duality of Tracking Problems 2.1, 2.2 and Control Problems 3.1, 3.2 means that the methods of solution and structure of the solutions of the latter problems discussed in detail in [8, 11] are also valid for the former (tracking) problems.

Note 4.2. Let $\varepsilon = 0$ in Problem 2.1. The phase restrictions (3.3) ((3.4)) in conjugate Control Problem 3.1 are then replaced by the condition $B_1 y(\tau) \equiv 0$, $t \leqslant \tau \leqslant \vartheta$, which means that the trajectory $y(\tau)$ must remain in a given subspace E_n.

5. Tracking in system with lag. The tracking problem in systems with lag can be formulated in various ways [14, 15]. We shall consider a problem very similar to those analyzed in Sects. 1–4.

Suppose we are given the n-dimensional system

$$dx(\tau) / d\tau = Ax(\tau) + Gx(\tau - h) \tag{5.1}$$

with the constant $n \times n$-matrices A, G and with a constant lag $h > 0$. Let the signal

$$z(\tau) = Nx(\tau) \qquad (t \leqslant \tau \leqslant \vartheta) \tag{5.2}$$

be detected in the interval $[t, \vartheta]$. Here N is a constant matrix of order $m \times n$.

We are to find an operation $\varphi[z]$ which isolates the linear combination $\zeta = c'x(\vartheta)$, where c is a given vector and ϑ a fixed number, on the basis of an arbitrary signal $z(\tau)$. As in Sect. 1, we seek the operation φ in the class $\varphi \in \Phi$, identifying the latter with the functions $v(\tau) \in V$.

Problem 5.1. (a) To find an operation $\varphi[z(t)]$, $\varphi \in \Phi$ satisfying the condition

$$\varphi[z] = c'x(\vartheta) \tag{5.3}$$

whatever the signal $z(\tau)$ defined by (5.2).

b) To find the optimal φ° of minimal norm among the solutions of (5.3).

In addition to $\varphi[z]$ we also seek an operation $\varphi_\xi[\tau]$ which isolates the quantity

$$\zeta(\xi) = c'x(\vartheta + \xi) = c'x_\vartheta(\xi), \qquad 0 \leqslant \xi \leqslant h$$

over the entire interval of length h on the basis of any signal $z(\tau)$, $t \leqslant \tau \leqslant \vartheta + \xi$. We identify the operation $\varphi_\xi[z]$ with the function $v(\xi, \tau)$, choosing

$$\varphi_\xi[z] \in \Phi^{(1)} \qquad (v(\xi, \tau) \in V^{(1)})$$

Here

$$\Phi^{(1)} = \bigcup_\nu \Phi_\nu^{(1)} = \{\varphi_\xi[z] : \|\varphi\|^{(1)} \leqslant \nu\} \qquad (\nu \geqslant 0)$$

The quantity $\|\varphi\|^{(1)}$ is a norm in the space of functions of two variables.

Problem 5.2. (a) To find an operation $\varphi_\xi[z(\tau)]$, $\varphi_\xi \in \Phi^{(1)}$ satisfying the condition

$$\varphi_\xi[z(\tau)] = c'x(\vartheta + \xi), \qquad 0 \leqslant \xi \leqslant h \tag{5.4}$$

whatever the signal $z(\tau)$ defined by Eq. (5.2) for $t \leqslant \tau \leqslant \vartheta + \xi$.

b) To find the optimal φ_ξ° of minimal norm from among the solutions of (5.3).

Let us consider Problem 5.1. Expressing the solution of (5.1) with the initial function $f(s)$, $t - h \leqslant s \leqslant t$ in integral form [16], we obtain

$$\varphi[z(\tau)] = \int_t^\vartheta v(\tau) N \Big(X(\tau, t) f(t) + \int_{t-h}^t X(\tau, s + h) Gf(s)\, ds\Big)\, d\tau$$

$$(X(\tau, \tau) = E;\ X(\vartheta, \tau) = 0,\ \vartheta < \tau)$$

Here $X(\vartheta, \tau)$ is a function matrix which satisfies Eq. (5.1) in ϑ and the conjugate equation

$$dX(\vartheta, \tau) / d\tau = -X(\vartheta, \tau)A - X(\vartheta, \tau + h)G$$

in τ.

On the other hand,

$$c'x(\vartheta) = c'\left(X(\vartheta, t) f(t) + \int_{t-h}^{t} X(\vartheta, s+h) Gf(s)\, ds\right)$$

By the condition of Problem 5.1 the equation $\varphi[z(\tau)] = c'x(\vartheta)$ must be fulfilled for any function $f(t+s)$, $-h \leqslant s \leqslant 0$. Thus, the condition

$$\left(\int_{t}^{\vartheta} v(\tau) NX(\tau, t)\, d\tau - c'X(\vartheta, t)\right) f(t) +$$

$$+ \int_{t-h}^{t} \left(\int_{t}^{\vartheta} v(\tau) NX(\tau, s+h) G d\tau - c'X(\vartheta, s+h) G\right) f(s)\, ds = 0$$

must be fulfilled for all n-vectors $f(t)$ and all n-vector functions $f(s)$, $t - h \leqslant s \leqslant t$. From this we immediately obtain the necessary and sufficient conditions of solvability of Problem 5.1, which are reducible to the equations

$$\int_{t}^{\vartheta} v(\tau) NX(\tau, t)\, d\tau = c'X(\vartheta, t) \tag{5.5}$$

$$\int_{t}^{\vartheta} v(\tau) NX(\tau, s) G d\tau = c'X(\vartheta, s) G \qquad (t \leqslant s \leqslant t+h) \tag{5.6}$$

Similarly, the solvability conditions for Problem 5.2 are equivalent to the equations

$$\int_{t}^{\vartheta+\xi} v(\xi, \tau) NX(\tau, t)\, d\tau = c'X(\vartheta+\xi, t) \tag{5.7}$$

$$\int_{t}^{\vartheta+\xi} v(\xi, \tau) NX(\tau, t+\eta) G d\tau = c'X(\vartheta+\xi, t+\eta) \tag{5.8}$$

$$(0 \leqslant \eta \leqslant h,\ 0 \leqslant \xi \leqslant h)$$

6. Control in systems with lag and lead. Let us consider the n-dimensional system

$$dy / d\tau = A_1 y(\tau) + G_1 y(\tau + h) + N_1 u \tag{6.1}$$

with the lead $h \geqslant 0$, the constant $n \times n$-matrices A_1, G_1, and the $n \times m$-matrix N. We choose the control u from the same class U as in Problem 3.1.

Problem 6.1. (a) The boundary condition

$$y(\tau) = f(\tau), \quad \vartheta \leqslant \tau \leqslant \vartheta + h$$

for system (5.1), where $f(\vartheta) = c$ and $f(\tau) = 0$ if $\tau > \vartheta$, is given. We are to find a control $u \in U$, $u(\tau) = 0$ for $\tau < t$ which takes system (6.1) from the state

$$y(\vartheta + \xi) = f(\vartheta + \xi) \quad (0 \leqslant \xi \leqslant h)$$

to the equilibrium state

$$y(t - \eta) \equiv 0 \quad (0 \leqslant \eta \leqslant \eta)$$

b) To find the optimal u° of minimal norm from among the solutions $u(\tau)$ of Item (a).

Our formulation of Problem (6.1) originates in [17]. Problem 6.1 (a) can be solved if and only if [18−20]

$$y(t) = 0, \qquad G_1 y(\tau) \equiv 0, \quad t \leqslant \tau \leqslant t + h \tag{6.2}$$

Writing out these conditions in detail, we obtain the equations

$$\int_{\vartheta}^{t} Y(t,\tau) N_1 u(\tau)\, d\tau + Y(t,\vartheta) c = 0 \tag{6.3}$$

$$G_1 \int_{\vartheta}^{t+\eta} Y(t+\eta,\tau) N_1 u(\tau)\, d\tau + G_1 Y(t+\eta,\vartheta) c = 0$$

Here $Y(t, \tau)$ satisfies Eq. (6.1) (for $u = 0$) and the equation with lag

$$dx(\tau) / d\tau = -x(\tau) A_1 - x(\tau - h) G_1$$

and the boundary condition

$$Y(t, \tau) \equiv 0, \quad t > \tau; \qquad Y(\tau, \tau) = E$$

in τ.

For

$$A_1 = -A', \qquad G_1 = -G' \tag{6.4}$$

we have

$$X(t, \tau) = Y'(\tau, t) \tag{6.5}$$

Problem 6.2. To find a function $u(\xi, \tau)$ which satisfies conditions (6.2), (6.3) with ϑ replaced by $\vartheta_1 = \vartheta + \xi$ for all $0 \leqslant \xi \leqslant h$.

Problem 6.2 essentially requires that Problem 6.1 be solvable with the variable ϑ_1 ($\vartheta \leqslant \vartheta_1 \leqslant \vartheta + h$) replacing the variable ϑ in the condition of Problem 6.1. Functions $u(\xi, \tau)$ of two variables are admissible in Problem 6.2, however.

Let problem (6.2), (6.3) be solved by the function $u(\tau)$. The function

$$u(\xi, \tau) = u(\xi + \tau) \qquad \vartheta + \xi \geqslant \tau \geqslant t + \xi$$

$$u(\xi, \tau) \equiv 0, \qquad \tau < t + \xi, \qquad \tau > \vartheta + \xi$$

then solves Problem 6.2. This can be verified directly by considering (6.1)–(6.3) with allowance for the stationarity of system (6.1).

The control problems (i.e. the analogs of Problem 6.1, 6.2) for the system with lag

$$dr(\tau) / d\tau = A_2 r(\tau) + G_2 r(\tau - h) + B_2 w \tag{6.6}$$

where r is the phase vector and w the control, can be considered in the same way.

Here, e.g. in the case of Problem 6.1, we are required to take $r(\tau)$ from the state

$$r_0(\tau) \equiv 0, \ \tau < t, \ r_0(t) = r_0$$

to the state

$$r(\tau) \equiv 0, \quad \vartheta \leqslant \tau \leqslant \vartheta + h$$

in the class of controls $w \in W$, $w(\tau) \equiv 0$ for $\tau > \vartheta$.

System (6.6) is obtainable from (6.1) for

$$A_2 = -A, \qquad G_2 = -G, \qquad B_2 = -B \tag{6.7}$$

by introducing inverse time.

7. The duality of control and tracking for systems with lag.

The fact that Eqs. (5.5), (5.6) are equivalent to (6.2), (6.3) under condition (6.5) implies the following statement.

Theorem 7.1. Let Problem 5.1 (a) on finding the quantity $c'x(\vartheta)$ from the signal $z(\tau)$ (5.2) detected in $[t, \vartheta]$ be considered for system (5.1) with lag. Problem 5.1 (a) is then equivalent to Problem 6.1 (a) on constructing for system (6.1) with lead a control $u(\tau)$, $u(\tau) \equiv 0$ for $\tau < t$, $\tau > \vartheta$ which takes the system from the state

$y(\vartheta) = c,\ y(\tau) \equiv 0$ for $\tau > \vartheta$ to the equilibrium state $y(\tau) \equiv 0,\ t-h \leqslant \tau \leqslant t$.

Note 7.1. Condition (6.7) enables us to formulate Duality Theorem 7.1 with system (6.1), (6.4) with lead replaced by equivalent system (6.6), (6.7) with lag. Theorem 7.1 can also be extended to Problem 6.2 and its analogs.

Note 7.2. We have formulated problems on the precise tracking of the quantity $c'x(\vartheta)$ and on the exact guidance of system (6.1) to the origin. Here, however, as in Sects. 1–4, we can consider the duality between problems on ε-trackability and ε-controllability, where the initial conditions must be satisfied to within ε only. Finally, it is possible to combine the arguments of Sects. 1–4 and 5–7.

Note 7.3. Theorem 7.1 implies that the property of complete (for any c) trackability of system (5.1), (5.2) is equivalent to the property of complete (for any c) controllability of system (6.1) from the state $\{y(\vartheta) = c,\ y(\tau) \equiv 0 \text{ for } \tau > \vartheta\}$ to the equilibrium position.

Note 7.4. If the controls $u(\tau)$ on Problem 6.1 (a) are chosen in the class of nth order distributions [3], then the sufficient conditions of complete controllability of system (6.1) are reducible to the general position condition [1] for the matrices A_1 and N_1.

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ON THE OPTIMAL STABILIZATION OF CONTROLLED SYSTEMS

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Stabilization of the steady motion of a system by additional forces with minimization of a certain functional characterizing control quality [1] is considered. The problem of determining the form of the integrand in the quality criterion and of the controlling forces from a certain class in such a way that the Liapunov function for the uncontrolled system can serve as the Liapunov function for the same system under the action of additional controlling forces is investigated. This problem is close to the inversion problem of analytical regulator construction [2]. The problem of optimal stabilization in some of the parameters [3] is stated and a theorem generalizing the basic theorem on stabilization in all the variables [1] is proved. Both problems are considered with specific reference to mechanical systems with a generalized energy integral of fixed sign. The results are illustrated by means of several examples. These include the problem of optimal stabilization of the positions relative to equilibrium and of the steady motions of a gyrostat satellite.

1. Let us consider the equations of perturbed motion of some system

$$\frac{dx_s}{dt} = X_s(t, x_1, \ldots, x_n) \qquad (s = 1, \ldots, n) \tag{1.1}$$

whose right sides X_s are defined in the domain

$$t \geqslant t_0, \quad |x_s| \leqslant H, \quad H = \text{const} > 0 \quad (s = 1, \ldots, n) \tag{1.2}$$

We assume that the functions X_s in domain (1.2) are continuous and that they satisfy the conditions which ensure the existence and uniqueness of the solutions of Eqs. (1.1) under any initial conditions from the domain (1.2); we also assume fulfillment of the